

Operator of Time and Generalized Schrödinger Equation

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Abstract

Within the framework of self-adjoint operator of time in non-relativistic quantum mechanics the equation describing change of the state of quantum system with respect to energy is introduced. The operator of time appears to be the generator of the change of energy while the operator of energy, that is conjugate to the operator of time, generates time evolution in this proposal. Two examples, one with discrete time and the other with continuous one, are given and the generalization of Schrödinger equation is proposed.

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The most important equation of quantum mechanics is, of course, the Schrödinger equation [1]. It is seen as equation that, for particular choice of Hamiltonian, determines how state of quantum mechanical system changes with respect to time. Usually, the Hamiltonian is a function of coordinate and momentum, $H(\hat{q}, \hat{p})$, but there are situations in which one is interested in time dependent Hamiltonians, as well [2-8]. In these cases, one can find how the energy of system changes in time, which is the consequence of the influence of environment on a system under consideration. However, the Schrödinger equation is not appropriate for finding an answer to question

how the state of quantum system changes with respect to the change of energy, acquired or lost by the system. In order to tackle this problem, one needs equation in which, so to say, one differentiates with respect to the energy, not with respect to the time as in the Schrödinger equation. This is what we are going to discuss in the present article and this will be done by using the formalism of the operator of time.

There is a whole variety of topics and approaches related to the operator of time, *e.g.*, [9-11] and references therein, but let us shortly review our approach that we have proposed in [12-14].

In order to fulfill demand coming from general relativity, that space and time should be treated on equal footing, just like for every spatial degree of freedom a separate Hilbert space is introduced, one should introduce Hilbert space where operator of time \hat{t} acts. More concretely, for the case of one degree of freedom, beside \hat{q} and conjugate momentum \hat{p} , acting non-trivially in \mathcal{H}_q , there should be \mathcal{H}_t where \hat{t} , together with \hat{s} that is conjugate to \hat{t} , act non-trivially. So, in $\mathcal{H}_q \otimes \mathcal{H}_t$ for the self-adjoint operators $\hat{q} \otimes \hat{I}$, $\hat{p} \otimes \hat{I}$, $\hat{I} \otimes \hat{t}$ and $\hat{I} \otimes \hat{s}$ the following commutation relations hold:

$$\frac{1}{i\hbar}[\hat{q} \otimes \hat{I}, \hat{p} \otimes \hat{I}] = \hat{I} \otimes \hat{I},$$

$$\frac{1}{i\hbar}[\hat{I} \otimes \hat{t}, \hat{I} \otimes \hat{s}] = -\hat{I} \otimes \hat{I}.$$

The other commutators vanish. The operator of time \hat{t} has continuous spectrum $\{-\infty, +\infty\}$, just like the operators of coordinate and momentum \hat{q} and \hat{p} . So is the case for the operator \hat{s} , that is conjugate to time, and which is the operator of energy. After noticing complete similarity among coordinate and momentum on one side and time and energy on the other side, one can introduce eigenvectors of \hat{t} :

$$\hat{t}|t\rangle = t|t\rangle, \quad \text{for every } t \in \mathbf{R}.$$

In $|t\rangle$ representation, operator of energy becomes $i\hbar \frac{\partial}{\partial t}$, while its eigenvectors $|E\rangle$ become $e^{\frac{1}{i\hbar}Et}$, for every $E \in \mathbf{R}$.

After Pauli, it is well known that there is no self-adjoint operator of time that is conjugate to the Hamiltonian $H(\hat{q}, \hat{p})$ which has bounded from below spectrum. Within our proposal, self-adjoint operator of time is conjugate to the operator of energy, which has unbounded spectrum. The Hamiltonian

and the operator of energy are acting in different Hilbert spaces, but there is subspace of the total Hilbert space where:

$$\hat{s}|\psi\rangle = H(\hat{q}, \hat{p})|\psi\rangle. \quad (1)$$

The states that satisfy this equation are physical since these states have non-negative energy. The last equation is nothing else but the Schrödinger equation. By taking $|q\rangle \otimes |t\rangle$ representation of previous equation, one gets the familiar form of Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \psi(q, t) = \hat{H} \psi(q, t), \quad (2)$$

with the shorthand notation $\hat{H} = \langle q|H(\hat{q}, \hat{p})|q'\rangle$. In other words, operator of energy has negative eigenvalues as well as non-negative, but the Schrödinger equation appears as a constraint that selects physically meaningful states, *i. e.*, states with non-negative energy, due to the non-negative spectrum of $H(\hat{q}, \hat{p})$, are selected by the Schrödinger equation. For the time independent Hamiltonian, typical solution of the Schrödinger equation (2) is $\psi_E(q)e^{\frac{1}{i\hbar}Et}$, which is $|q\rangle \otimes |t\rangle$ representation of $|\psi_E\rangle \otimes |E\rangle$, where $H(\hat{q}, \hat{p})|\psi_E\rangle = E|\psi_E\rangle$ and $\hat{s}|E\rangle = E|E\rangle$. For such states, the Heisenberg uncertainty relation for \hat{s} and \hat{t} obviously holds. The energy eigenvectors $|E\rangle$ have the same formal characteristics as, say, the momentum eigenvectors, *i. e.*, they are normalized to $\delta(0)$ and, for different values of energy, they are mutually orthogonal.

Therefore, the Schrödinger equation demands equal action of the operator of energy \hat{s} and Hamiltonian $H(\hat{q}, \hat{p})$ on states of quantum system and, on the other hand, due to the fact that the operator of energy in time representation is $i\hbar \frac{\partial}{\partial t}$, it is said for the Schrödinger equation that it describes how the state of a quantum system changes with time. In the Schrödinger equation the Hamiltonian appears as dynamical counterpart of energy. The Hamiltonian is often seen as the one that determines time evolution of states of quantum system since its eigenvalues appear in phase factor $\langle t|E\rangle = e^{\frac{1}{i\hbar}Et}$.

In analogy with this, one can introduce:

$$\hat{t}|\psi\rangle = G(\hat{q}, \hat{p})|\psi\rangle. \quad (3)$$

By taking $|p\rangle \otimes |E\rangle$ representation of previous equation, one gets:

$$-i\hbar \frac{\partial}{\partial E} \psi(p, E) = \hat{G} \psi(p, E), \quad (4)$$

With the shorthand notation $\hat{G} = \langle p|G(\hat{q}, \hat{p})|p'\rangle$. (Instead of $|p\rangle$ representation one can, of course, use $|q\rangle$ representation. The momentum representation is taken just because energy and momentum form quadri vector. The similar remark applies for the above $|q\rangle \otimes |t\rangle$ representation.)

In (3) one demands that the operator of time and its dynamical counterpart $G(\hat{q}, \hat{p})$ should equally act on states of quantum mechanical system. As the original Schrödinger equation, this equation represents constraint, as well. The original Schrödinger equation can be seen as the energy constraint, while this one is time constraint that also selects subspace in $\mathcal{H}_q \otimes \mathcal{H}_t$. After representing (3) in $|p\rangle \otimes |E\rangle$ basis, in analogy with the interpretation of original Schrödinger equation, it could be said that this equation describes how the state of quantum mechanical system changes with respect to the change of energy. Just like the time is treated as evolution parameter for (2), the energy in (4) can be seen as independent variable.

The typical solution of, let us call it, the second Schrödinger equation (4) is $\psi_t(p)e^{\frac{-1}{i\hbar}Et}$. It is $|p\rangle \otimes |E\rangle$ representation of $|\psi_t\rangle \otimes |t\rangle$, where $G(\hat{q}, \hat{p})|\psi_t\rangle = t|\psi_t\rangle$ and $\hat{t}|t\rangle = t|t\rangle$. For such states, the Heisenberg uncertainty relation for \hat{s} and \hat{t} obviously holds.

Let us stress that the generators of the Lie algebra (beside coordinate and momentum) are \hat{s} and \hat{t} , and not $H(\hat{q}, \hat{p})$ and $G(\hat{q}, \hat{p})$. For the former it holds $\frac{1}{i\hbar}[\hat{t}, \hat{s}] = -\hat{I}$, while for the latter ones, since they are just dynamical counterparts of the energy and time, *a priori* there is no reason to demand non-commutativity, *i. e.*, their commutator depends on particular choice of these functions of coordinate and momentum.

As for different systems are appropriate different Hamiltonians, for different situations one should use appropriate $G(\hat{q}, \hat{p})$. As an illustration, let us briefly discuss two examples of $G(\hat{q}, \hat{p})$. Due to the Big Bang as the beginning of time, it seems reasonable to assume $G(\hat{q}, \hat{p})$ with bounded from below spectrum. Without going into debate about existence of time crystals, the first example offers a toy model of discrete time while for the second one the quantum system is characterized with the continuous time.

If $G(\hat{q}, \hat{p})$ is:

$$G(\hat{q}, \hat{p}) = \frac{\hbar}{m^2 c^4} \left(\frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega \hat{q}^2 \right),$$

then solutions of the second Schrödinger equation (3) are $|\psi_n\rangle \otimes |t_n\rangle$, $n \in \mathbb{N}$, where $|\psi_n\rangle$ are well known solutions of the eigenvalue problem for Hamilto-

nian of harmonic oscillator and:

$$t_n = \frac{\hbar^2 \omega}{m^2 c^4} \left(n + \frac{1}{2}\right).$$

If the standard ladder operators \hat{a}^\dagger and \hat{a} , that act on $|\psi_{t_n}\rangle$, are directly multiplied by time translation operator $\hat{a}^\dagger \otimes e^{\frac{1}{i\hbar}\hat{s}\Delta t}$ and $\hat{a} \otimes e^{\frac{1}{i\hbar}\hat{s}\Delta t}$, where $\Delta t = \frac{\hbar^2 \omega}{m^2 c^4}$, then it holds:

$$\hat{a}^\dagger \otimes e^{-\frac{1}{i\hbar}\hat{s}\Delta t} |\psi_{t_n}\rangle \otimes |t_n\rangle = \sqrt{n+1} |\psi_{t_{n+1}}\rangle \otimes |t_{n+1}\rangle,$$

$$\hat{a} \otimes e^{\frac{1}{i\hbar}\hat{s}\Delta t} |\psi_{t_n}\rangle \otimes |t_n\rangle = \sqrt{n} |\psi_{t_{n-1}}\rangle \otimes |t_{n-1}\rangle.$$

So, these operators formally describe transition between states that represent nearby moments in time. Distinguished moments t_n are the ones in which system can be located in time.

The unitary operator attached to change of the energy is $e^{\frac{1}{i\hbar}\hat{t}\Delta E}$. *Nota bene*, the unitary operator that represents time evolution is $e^{\frac{1}{i\hbar}\hat{s}\Delta t}$, not the $e^{\frac{1}{i\hbar}H(\hat{q},\hat{p})\Delta t}$, because \hat{s} is the generator of time translation in \mathcal{H}_t , and not $H(\hat{q},\hat{p})$ since it does not even act in \mathcal{H}_t . Of course, these two unitary operators are effectively the same when they act on solutions of the original Schrödinger equation.

Another example is the case with continuous time, when $G(\hat{q},\hat{p}) = \frac{\hbar}{m^3 c^4} \hat{p}^2$. The solutions of the second Schrödinger equation are then $|p\rangle \otimes |t\rangle$, where $t = \frac{\hbar}{m^3 c^4} p^2$. This $G(\hat{q},\hat{p})$ commutes with the Hamiltonian of a free particle, while the previous one obviously commutes with the Hamiltonian of harmonic oscillator. Needless to say, in general, one can combine every $H(\hat{q},\hat{p})$ with any $G(\hat{q},\hat{p})$. However, with particular $H(\hat{q},\hat{p})$ and $G(\hat{q},\hat{p})$ one defines some physical system for which, according to the standard interpretation, it is possible to describe changes in time and energy. By staying with the standard interpretation, one can say that if some quantum system evolves under the action of $H(\hat{q},\hat{p})$ until the moment, say, t_0 , when it changes energy, then one should apply $\hat{U}(i,j) \otimes e^{\frac{1}{i\hbar}\hat{t}\Delta E}$ on state $|E_i\rangle \otimes |E_i\rangle$, that is solution of the original Schrödinger equation for E_i , and get the other solution $|E_j\rangle \otimes |E_j\rangle$ of the same equation (here $\hat{U}(i,j)$ is the unitary operator that rotates $|E_i\rangle$ into $|E_j\rangle$ in the first Hilbert space and ΔE is the energy difference among given states). The moment t_0 at which this can happen depends on $G(\hat{q},\hat{p})$, *i. e.*, it has to belong to the spectrum of $G(\hat{q},\hat{p})$, just like E_i belongs to the spectrum of

$H(\hat{q}, \hat{p})$. In other words, $H(\hat{q}, \hat{p})$ and $G(\hat{q}, \hat{p})$ determine what are the solutions of original and second Schrödinger equations, and everything related to this. In particular, on $H(\hat{q}, \hat{p})$ and $G(\hat{q}, \hat{p})$ depend in which amounts energy of the system can be changed and in what time intervals this can happen if there is a sequence of time evolutions followed by the changes of the systems energy.

This leads us to the generalization of the original Schrödinger equation:

$$(c_s \hat{s} + c_t \hat{t})|\Psi\rangle = F(\hat{q}, \hat{p}, \hat{s}, \hat{t})|\Psi\rangle, \quad |\Psi\rangle \in \mathcal{H}_q \otimes \mathcal{H}_t. \quad (5)$$

This equation superposes both types of the systems change. Obviously, the original and second Schrödinger equation follow from (5) for the adequate choices of dimensional constants c_s , c_t and $F(\hat{q}, \hat{p}, \hat{s}, \hat{t})$. This equation is appropriate for investigations of situations with time dependant Hamiltonians, as well. However, such Hamiltonians shall be considered elsewhere.

Finally, let us briefly comment measurement outcomes of energy and time. Within the present formalism, both \hat{s} and \hat{t} have the whole \mathbb{R} as spectrum, but only the values determined by $H(\hat{q}, \hat{p})$ and $G(\hat{q}, \hat{p})$ through Schrödinger equations can be attributed to the system under consideration. In other words, the physical system is defined by $H(\hat{q}, \hat{p})$ and $G(\hat{q}, \hat{p})$ and Schrödinger equations select values of energy and time that are characteristic to that system. While other values are formally possible, they are unrelated to the system. Therefore, description of measurement process should use orthogonal resolution of identity operator in a subspace determined by (1) or (3), not the identity operator in whole $\mathcal{H}_q \otimes \mathcal{H}_t$.

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